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MORE ON THE LINEARIZATION OF W-ALGEBRAS

S. Krivonos* and A. Sorin†

Bogoliubov Laboratory of Theoretical Physics, JINR,
141980, Dubna, Moscow Region, Russia

Abstract

We show that a wide class of W-(super)algebras, including $W_N^{(N-1)}$, $U(N)$-superconformal as well as W_N nonlinear algebras, can be linearized by embedding them as subalgebras into some *linear* (super)conformal algebras with finite sets of currents. The general construction is illustrated by the example of W_4 algebra.

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*E-mail: krivonos@thsun1.jinr.dubna.su
†E-mail: sorin@thsun1.jinr.dubna.su

1 Introduction

Since the pioneer paper of Zamolodchikov [1], a lot of extended nonlinear conformal algebras (the W-type algebras) have been constructed and studied (see, e.g., [2] and references therein). The growing interest to this subject is motivated by many interesting applications of nonlinear algebras to the string theory, integrable systems, etc. However, the intrinsic nonlinearity of W-algebras makes it rather difficult to apply to them the standard arsenal of techniques and means used in the case of linear algebras (while constructing their field representations, etc.). A way to circumvent this difficulty has been proposed by us in [3]. We found that in many cases a given nonlinear W algebra can be embedded into some linear conformal algebra which is generated by a finite number of currents and contains the considered W-algebra as subalgebra in some nonlinear basis. Up to now the explicit construction has been carried out for some simplest examples of nonlinear (super)algebras (W_3 and $W_3^{(2)}$ [3], WB_2 and $W_{2,4}$ [4]). Besides being a useful tool to construct new field realizations of nonlinear algebras [3-4], these linear algebras provide a suitable framework for considering the embeddings of the Virasoro string in the W-type ones [5].

In the present letter[1] we show that the linearization is a general property inherent to many nonlinear W-type algebras. We demonstrate that a wide class of W-(super)algebras, including $U(N)$-superconformal [6], $W_N^{(N-1)}$ [7-9], as well as W_N [10] algebras, admit a linearization. The explicit formulas related linear and nonlinear algebras for all these cases are given. The example of W_4 algebra is elaborated in detail.

2 Linearizing $U(N)$ (quasi)superconformal algebras.

In this Section we construct linear conformal algebras which contain the algebra $W_{N+2}^{(N+1)}$ or $U(N)$ superconformal algebras as subalgebras in some nonlinear basis. By this we mean, that the currents of nonlinear algebras can be related by an *invertible* transformation to those of linear algebras. In what follows these linear algebras will be called the linearizing algebras for nonlinear ones.

Let us start by reminding the operator product expansions (OPE's) for the $W_{N+2}^{(N+1)}$ algebras and $U(N)$ superconformal algebras (SCA). The OPE's for these algebras can be written in a general uniform way keeping in mind that the $W_N^{(N-1)}$ algebra is none other than $U(N-2)$ quasi-superconformal algebra (QSCA) [7-9] [2]. Both $U(N)$ SCA and $U(N)$ QSCA have the same number of generating currents: the stress tensor $T(z)$, the $U(1)$ current $U(x)$, the $SU(N)$ Kac-Moody currents $J_a^b(x)$ $(1 \leq a, b \leq N, \mathrm{Tr}(J) = 0)$ and two sets of currents in the fundamental $G_a(x)$ and conjugated $\bar{G}^b(x)$ representations of $SU(N)$. The currents $G_a(x), \bar{G}^b(x)$ are bosonic for $U(N)$ QSCA and fermionic for $U(N)$ SCA. To distinguish between these two cases we, following refs. [8], introduce the parameter ϵ equal to $1(-1)$ for the QSCAs (SCAs) and write the OPE's for these algebras in the following universal form:

$$T(z_1)T(z_2) \;=\; \frac{c/2}{z_{12}^4} + \frac{2T}{z_{12}^2} + \frac{T'}{z_{12}} \quad , \quad U(z_1)U(z_2) = \frac{c_1}{z_{12}^2} \; ,$$

[1] The preliminary version of this Letter has been present as talk at the International Workshop "Finite Dimensional Integrable Systems", July 18-21, JINR, Dubna, 1994.

[2] Strictly speaking, the $W_N^{(N-1)}$ algebra coincides with $GL(N-2)$ QSCA. In what follows, we will not specify the real forms of algebras and use the common term $U(N)$ QSCA.

$$
\begin{aligned}
T(z_1)J_a^b(z_2) &= \frac{J_a^b}{z_{12}^2}+\frac{J_a^{b\prime}}{z_{12}} \quad , \quad T(z_1)U(z_2)=\frac{U}{z_{12}^2}+\frac{U'}{z_{12}} \, , \\
T(z_1)G_a(z_2) &= \frac{3/2G_a}{z_{12}^2}+\frac{G_a}{z_{12}} \quad , \quad T(z_1)\bar{G}^a(z_2)=\frac{3/2\bar{G}^a}{z_{12}^2}+\frac{\bar{G}^a}{z_{12}} \, , \\
J_a^b(z_1)J_c^d(z_2) &= (K-\epsilon-N)\frac{\delta_a^d\delta_c^b-\frac{1}{N}\delta_a^b\delta_c^d}{z_{12}^2}+\frac{\delta_c^bJ_a^d-\delta_a^dJ_c^b}{z_{12}} \, , \\
U(z_1)G_a(z_2) &= \frac{G_a}{z_{12}} \quad , \quad U(z_1)\bar{G}^a(z_2)=-\frac{\bar{G}^a}{z_{12}} \, , \\
J_a^b(z_1)G_c(z_2) &= \frac{\delta_c^bG_a-\frac{1}{N}\delta_a^bG_c}{z_{12}} \quad , \quad J_a^b(z_1)\bar{G}^c(z_2)=\frac{-\delta_a^c\bar{G}^b+\frac{1}{N}\delta_a^b\bar{G}^c}{z_{12}} \\
G_a(z_1)\bar{G}^b(z_2) &= \frac{2\delta_a^bc_2}{z_{12}^3}+\frac{2x_2\delta_a^bU+2x_3J_a^b}{z_{12}^2}+\frac{x_2\delta_a^bU'+x_3J_a^{b\prime}+2x_5(J_a^dJ_d^b)}{z_{12}}+ \\
&\quad \frac{2x_4(UJ_a^b)+\delta_a^b\left(x_1(UU)-2\epsilon T+2x_6(J_d^eJ_e^d)\right)}{z_{12}} \, , \qquad (2.1)
\end{aligned}
$$

where the central charges c and parameters x are defined by

$$
\begin{aligned}
c &= \frac{-6\epsilon K^2+(N^2+11\epsilon N+13)K-(\epsilon+N)(N^2+5\epsilon N+6)}{K} \, , \\
c_1 &= \frac{N(2K-N-2\epsilon)}{2+\epsilon N} \quad , \quad c_2=\frac{(K-N-\epsilon)(2K-N-2\epsilon)}{K} \, , \\
x_1 &= \frac{(\epsilon+N)(2\epsilon+N)}{N^2K} \quad , \quad x_2=\frac{(2\epsilon+N)(K-\epsilon-N)}{\epsilon NK} \quad , \quad x_3=\frac{2K-N-2\epsilon}{K} \, , \\
x_4 &= \frac{2+\epsilon N}{NK} \quad , \quad x_5=\frac{1}{K} \quad , \quad x_6=\frac{1}{2\epsilon K} \, . \qquad (2.2)
\end{aligned}
$$

The currents in the r.h.s. of OPE's (2.1) are evaluated at the point z_2, $z_{12}=z_1-z_2$ and the normal ordering in the nonlinear terms is understood.

The main question we need to answer in order to linearize the algebras (2.1) is as to which minimal set of additional currents must be added to (2.1) to get extended linear conformal algebras containing (2.1) as subalgebras. The idea of our construction comes ¿from the observation that the classical ($K\to\infty$) $U(N)$ (Q)SCA (2.1) can be realized as left shifts in the following coset space

$$
g=e^{\int dz\bar{Q}^a(z)G_a(z)} \, , \qquad (2.3)
$$

which is parametrized by N parameters-currents $\bar{Q}^a(z)$ with unusual conformal weights $-1/2$. In this case, all the currents of $U(N)$ (Q)SCA (2.1) can be constructed from $\bar{Q}^a(z)$, their conjugated momenta $G_a(z)=\delta/\delta\bar{Q}^a$ and the currents of the maximal *linear* subalgebra $\mathcal{H}_N$

$$
\mathcal{H}_N=\left\{T,U,J_a^b,\bar{G}^a\right\} \, . \qquad (2.4)
$$

Though the situation in quantum case is more difficult, it seems still reasonable to try to extend the $U(N)$ (Q)SCA (2.1) by N additional currents $\bar{Q}^a(z)$ with conformal weights $-1/2$.[3]

[3]Let us remind that the current with just this conformal weight appears in the linearization of $W_3^{(2)}$ algebra [3].

Fortunately, this extension is sufficient to construct the linearizing algebras for the $U(N)$ (Q)SCAs. Without going into details, let us write down the set of OPE's for these linear algebras, which we will denote as $(Q)SCA_N^{lin}$

$$
\begin{aligned}
T(z_1)T(z_2) &= \frac{c/2}{z_{12}^4}+\frac{2T}{z_{12}^2}+\frac{T'}{z_{12}} \quad , \quad U(z_1)U(z_2)=\frac{c_1}{z_{12}^2}\ , \\
T(z_1)J_a^b(z_2) &= \frac{J_a^b}{z_{12}^2}+\frac{J_a^{b\prime}}{z_{12}} \quad , \quad T(z_1)U(z_2)=\frac{U}{z_{12}^2}+\frac{U'}{z_{12}}\ , \\
T(z_1)G_a(z_2) &= \frac{3/2G_a}{z_{12}^2}+\frac{G_a}{z_{12}} \quad , \quad T(z_1)\bar{\widetilde{G}}^a(z_2)=\frac{3/2\bar{\widetilde{G}}^a}{z_{12}^2}+\frac{\bar{\widetilde{G}}^a}{z_{12}}\ , \\
T(z_1)\bar{Q}^a(z_2) &= \frac{-1/2\bar{Q}^a}{z_{12}^2}+\frac{\bar{Q}^a}{z_{12}}\ , \\
J_a^b(z_1)J_c^d(z_2) &= (K-\epsilon-N)\frac{\delta_a^d\delta_c^b-\frac{1}{N}\delta_a^b\delta_c^d}{z_{12}^2}+\frac{\delta_c^bJ_a^d-\delta_a^dJ_c^b}{z_{12}}\ , \\
U(z_1)G_a(z_2) &= \frac{G_a}{z_{12}} \quad , \quad U(z_1)\bar{\widetilde{G}}^a(z_2)=-\frac{\bar{\widetilde{G}}^a}{z_{12}} \quad , \quad U(z_1)\bar{Q}^a(z_2)=-\frac{\bar{Q}^a}{z_{12}}\ , \\
J_a^b(z_1)G_c(z_2) &= \frac{\delta_c^bG_a-\frac{1}{N}\delta_a^bG_c}{z_{12}} \quad , \quad J_a^b(z_1)\bar{\widetilde{G}}^c(z_2)=\frac{-\delta_a^c\bar{\widetilde{G}}^b+\frac{1}{N}\delta_a^b\bar{\widetilde{G}}^c}{z_{12}}\ , \\
J_a^b(z_1)\bar{Q}^c(z_2) &= \frac{-\delta_a^c\bar{Q}^b+\frac{1}{N}\delta_a^b\bar{Q}^c}{z_{12}}\ , \\
G_a(z_1)\bar{Q}^b(z_2) &= \frac{\delta_a^b}{z_{12}} \quad , \quad G_a(z_1)\bar{\widetilde{G}}^b(z_2)=\text{regular}\ .
\end{aligned}
\tag{2.5}
$$

Here the central charges c and c_1 are the same as in (2.2) and the currents $G_a(z)$, $\bar{\widetilde{G}}^a(z)$ and $\bar{Q}^a(z)$ are bosonic (fermionic) for $\epsilon=1(-1)$.

In order to prove that the linear algebra $(Q)SCA_N^{lin}$ (2.5) contains $U(N)$ (Q)SCA (2.1) as a subalgebra, let us perform the following *invertible* nonlinear transformation to the new basis $\left\{T(z),U(z),J_a^b(z),G_a(z),\bar{G}^a(z),\bar{Q}^a(z)\right\}$, where the "new" current $\bar{G}^a(z)$ is defined as

$$
\begin{aligned}
\bar{G}^a = {} & \bar{\widetilde{G}}^a+y_1\bar{Q}^{a\prime\prime}+y_2(J_b^a\bar{Q}^{b\prime})+y_3(U\bar{Q}^{a\prime})+y_4(J_b^{a\prime}\bar{Q}^b)+y_5(U'\bar{Q}^a)+y_6(T\bar{Q}^a)+ \\
& y_7(J_b^cJ_c^a\bar{Q}^b)+y_8(J_b^cJ_c^b\bar{Q}^a)+y_9(UJ_b^a\bar{Q}^b)+y_{10}(UU\bar{Q}^a)+y_{11}(J_b^cG_c\bar{Q}^b\bar{Q}^a)+ \\
& y_{12}(J_b^aG_c\bar{Q}^c\bar{Q}^b)+y_{13}(G_b'\bar{Q}^b\bar{Q}^a)+y_{14}(G_b\bar{Q}^{b\prime}\bar{Q}^a)+y_{15}(G_b\bar{Q}^b\bar{Q}^{a\prime})+ \\
& y_{16}(G_bG_c\bar{Q}^b\bar{Q}^c\bar{Q}^a)+y_{17}(UG_b\bar{Q}^b\bar{Q}^a)\ ,
\end{aligned}
\tag{2.6}
$$

and the coefficients y_1-y_{17} are defined as

$$
\begin{aligned}
y_1 &= 2K \quad , \quad y_2=4 \quad , \quad y_3=\frac{2(2+\epsilon N)}{N} \quad , \quad y_4=\frac{2(K-\epsilon-N)}{K}\ , \\
y_5 &= \frac{(K-\epsilon-N)(2+\epsilon N)}{NK} \quad , \quad y_6=-2\epsilon \quad , \quad y_7=\frac{2}{K} \quad , \quad y_8=\frac{2}{\epsilon K} \quad , \quad y_9=\frac{2(2+\epsilon N)}{NK}\ , \\
y_{10} &= \frac{(\epsilon+N)(2\epsilon+N)}{N^2K} \quad , \quad y_{11}=y_{12}=\frac{2}{K} \quad , \quad y_{13}=\frac{2(K-N-2\epsilon)}{K} \quad , \quad y_{14}=4 \\
y_{15} &= 2 \quad , \quad y_{16}=\frac{2}{\epsilon K} \quad , \quad y_{17}=\frac{2(2+\epsilon N)}{NK}\ .
\end{aligned}
\tag{2.7}
$$

Now it is a matter of straightforward (though tedious) calculation to check that OPE's for the set of currents $\left\{T(z), U(z), J_a^b(z), G_a(z)\right\}$ and $\bar{G}^a(z)$ (2.6) coincide with the basic OPE's of the $U(N)$ (Q)SCA (2.1).

Thus, we have shown that the linear algebra $(Q)SCA_N^{lin}$ (2.5) contains $U(N)$ (Q)SCA as a subalgebra in the nonlinear basis.

We close this Section with a few comments.

First of all, we would like to stress that the pairs of currents $G_a(z)$ and $\bar{Q}^a(z)$ (with conformal weights equal to 3/2 and $-1/2$, respectively) in (2.5) look like "ghost–anti-ghost" fields and so $(Q)SCA_N^{lin}$ algebra (2.5) can be simplified by means of the standard ghost decoupling transformations

$$
\begin{aligned}
U &= \widetilde{U} - \epsilon(G_a\bar{Q}^a) \, , \\
J_a^b &= \widetilde{J}_a^b - \epsilon(G_a\bar{Q}^b) + \delta_a^b \frac{\epsilon}{N}(G_c\bar{Q}^c) \, , \\
T &= \widetilde{T} + \frac{1}{2}\epsilon(G_a'\bar{Q}^a) + \frac{3}{2}\epsilon(G_a\bar{Q}^{a\prime}) - \frac{\epsilon(2+\epsilon N)}{2K}\widetilde{U}' \, . \qquad (2.8)
\end{aligned}
$$

In the new basis the algebra $(Q)SCA_N^{lin}$ splits into the direct product of the ghost–anti-ghost algebra $\Gamma_N = \left\{\bar{Q}^a, G_b\right\}$ with the OPE's

$$
G_a(z_1)\bar{Q}^b(z_2) = \frac{\delta_a^b}{z_{12}}
$$

and the algebra of the currents $\left\{\widetilde{T}, \widetilde{U}, \widetilde{J}_a^b, \widetilde{\bar{G}}^a\right\}$. We denote the latter as $\widetilde{(Q)SCA}_N^{lin}$. It is defined by the following set of OPE's

$$
\begin{aligned}
\widetilde{T}(z_1)\widetilde{T}(z_2) &= \frac{-6\epsilon K^2 + (N^2+13)K - (N^3 - N + 6\epsilon)}{2K\, z_{12}^4} + \frac{2\widetilde{T}}{z_{12}^2} + \frac{\widetilde{T}'}{z_{12}} \, , \\
\widetilde{U}(z_1)\widetilde{U}(z_2) &= \left(\frac{2NK}{2+\epsilon N}\right)\frac{1}{z_{12}^2} \, , \; \widetilde{T}(z_1)\widetilde{J}_a^b(z_2) = \frac{\widetilde{J}_a^b}{z_{12}^2} + \frac{\widetilde{J}_a^{b\prime}}{z_{12}} \, , \\
\widetilde{T}(z_1)\widetilde{U}(z_2) &= \frac{\widetilde{U}}{z_{12}^2} + \frac{\widetilde{U}'}{z_{12}} \, , \\
\widetilde{T}(z_1)\widetilde{\bar{G}}^a(z_2) &= \left(\frac{3}{2} + \frac{\epsilon(2+\epsilon N)}{2K}\right)\frac{\widetilde{\bar{G}}^a}{z_{12}^2} + \frac{\widetilde{\bar{G}}^a}{z_{12}} \, , \\
\widetilde{J}_a^b(z_1)\widetilde{J}_c^d(z_2) &= (K-N)\frac{\delta_a^d\delta_c^b - \frac{1}{N}\delta_a^b\delta_c^d}{z_{12}^2} + \frac{\delta_c^b\widetilde{J}_a^d - \delta_a^d\widetilde{J}_c^b}{z_{12}} \, , \\
\widetilde{U}(z_1)\widetilde{\bar{G}}^a(z_2) &= -\frac{\widetilde{\bar{G}}^a}{z_{12}} \, , \; \widetilde{J}_a^b(z_1)\widetilde{\bar{G}}^c(z_2) = \frac{-\delta_a^c\widetilde{\bar{G}}^b + \frac{1}{N}\delta_a^b\widetilde{\bar{G}}^c}{z_{12}} \, , \\
\widetilde{\bar{G}}^a(z_1)\widetilde{\bar{G}}^b(z_2) &= \text{regular} \, , \qquad (2.9)
\end{aligned}
$$

$$
(Q)SCA_N^{lin} = \Gamma_N \otimes \widetilde{(Q)SCA}_N^{lin} \quad . \qquad (2.10)
$$

Secondly, note that the linear algebra $\widetilde{(Q)SCA}_N^{lin}$ (2.9) has the same number of currents and the same structure relations as the maximal linear subalgebra $\mathcal{H}_N$ (2.4) of $U(N)$ (Q)SCA (2.1),

but with the "shifted" central charges and conformal weights. It is of importance that the central charges and conformal weights are strictly related as in (2.9).[4] Otherwise, with another relation between these parameters, we would never find the $U(N)$ (Q)SCA (2.1) in $(Q)SCA_N^{lin}$. Thus, our starting assumption about the structure of linear algebra for $U(N)$ (Q)SCA coming from the classical coset realization approach, proved to be correct, modulo shifts of central charges and conformal weights.

Thirdly, let us remark that among the $U(N)$ (Q)SCAs there are many (super)algebras which are well known under other names. For examples:[5]

$$
\begin{aligned}
(Q)SCA(\epsilon=1,N=1) &\equiv W_3^{(2)} \quad [11], \\
(Q)SCA(\epsilon=-1,N=1) &\equiv N=2\; SCA \quad [12], \\
(Q)SCA(\epsilon=-1,N=2) &\equiv N=4\; SU(2)\; SCA \quad [12].
\end{aligned}
$$

Finally, let us remind that in the simplest case of $W_3^{(2)}$ algebra [3], the linear $Q\widetilde{SC}A_1^{lin}$ algebra (2.9) coincides with the linear algebra W_3^{lin} for W_3. For general N the situation is more complicated. This will be discussed in the next Section.

3 Linearizing W algebras.

The problem of construction of linear algebras for nonlinear ones can be naturally divided in two steps. As the first step we need to find the appropriate sets of additional currents which linearize the given nonlinear algebra. In other words, we must construct the linear algebra (like $(Q)SCA_N^{lin}$) with the correct relations between all central charges and conformal weights, which contains the nonlinear algebra as a subalgebra in some nonlinear basis. As the second step, we need to explicitly construct the transformation from the linear basis to a nonlinear one (like (2.6)). While the first step is highly non-trivial, the second one is purely technical. In principle, we could write down the most general expression with arbitrary coefficients and appropriate conformal weights, and then fix all the coefficients from the OPE's of the nonlinear algebra.

In this Section we will demonstrate that the linear algebra $QSCA_N^{lin}$ (2.5) constructed in the previous Section gives us the hints how to find the linear algebras for many other W-type algebras which can be obtained from the $GL(N)$ QSCAs via the secondary Hamiltonian reduction [13].

3.1 Secondary linearization.

The bosonic $GL(N)$ QSCAs (or, in another notation, $W_{N+2}^{(N+1)}$), which have been linearized in the previous Section, can be obtained through the Hamiltonian reduction ¿from the affine $sl(N+2)$

[4]Let us remark that Jacoby identities for the set of currents $\left\{\widetilde{T},\widetilde{U},\widetilde{J}_a^b,\widetilde{\widetilde{G}}^a\right\}$ do not fix neither central charges nor the conformal weight of $\widetilde{\widetilde{G}}^a$.

[5]To avoid the singularity in (2.2) at $\epsilon=-1,N=2$ one should firstly rescale the current $U\rightarrow\frac{1}{\sqrt{2+\epsilon N}}U$ and then put $\epsilon=-1,N=2$ [6].

algebras [7-9]. The constraints on the currents of $sl(N+2)$ algebra which yield $W_{N+2}^{(N+1)}$ read

$$\left(\begin{array}{cc|cccc} U & T & \overline{G}^1 & \overline{G}^2 & \ldots & \overline{G}^N \\ 1 & 0 & 0 & 0 & \ldots & 0 \\ \hline 0 & G_1 & & & & \\ 0 & G_2 & & & & \\ \vdots & \vdots & & sl(N) - \frac{\delta_a^b}{N} U & & \\ 0 & G_N & & & & \end{array}\right) \tag{3.1}$$

The $W_{N+2}^{(N+1)}$ algebras, forming in themselves a particular class of W-algebras with quadratic nonlinearity, are at the same time universal in the sense that a lot of other W-algebras can be obtained from them via the secondary Hamiltonian reduction (e.g., W_N algebras, etc.)[13].

Let us consider a set of possible secondary reductions of $W_{N+2}^{(N+1)}$ algebra (3.1). These are introduced by imposing the constraints

$$G_1 = 1 \quad , \quad G_2 = \ldots = G_N = 0 \quad , \tag{3.2}$$

$$sl(N)|_{sl(2)} \quad , \tag{3.3}$$

where we denoted as $sl(N)|_{sl(2)}$ the set of constraints on the $sl(N)$ currents, associated with an arbitrary embedding of $sl(2)$ algebra into $sl(N)$ subalgebra of $W_{N+2}^{(N+1)}$.

The main conjecture we will keep to in this Section is as follows

> *To find the linearizing algebra for a given nonlinear W-algebra related to $W_{N+2}^{(N+1)}$ through the Hamiltonian reduction (3.2),(3.3), one should apply the reduction (3.3) to the linear algebra $Q\widetilde{SCA}_N^{lin}$ (2.9) and then linearize the resulting algebra. The algebra $Q\widetilde{SCA}_N^{lin}$ itself is the linearizing algebra for the reduction (3.2).*

Roughly speaking, we propose to replace the linearization of the algebra W obtained from the nonlinear algebra $W_{N+2}^{(N+1)}$ through the full set of the Hamiltonian reduction constraints (3.2)-(3.3), by the linearization of the algebra $\widetilde{W}$ obtained from the *linear* algebra $Q\widetilde{SCA}_N^{lin}$ by imposing the relaxed set (3.3).

At present, we are not aware of the rigorous proof of this statement, but it works well both in the classical cases (on the level of Poisson brackets) and in many particular quantum examples. Of course, the secondary Hamiltonian reduction (3.3), being applied to $Q\widetilde{SCA}_N^{lin}$, gives rise to a nonlinear algebra. However, the problem of its linearization can be reduced to the linearization of reduction (3.3) applied to the affine algebra $sl(N) \subset Q\widetilde{SCA}_N^{lin}$, which was constructed in [14]. The resulting algebra will be just linear algebra for the nonlinear algebra we started with.

Let us briefly discuss the explicit construction of the linear algebra W^{lin} which contains the nonlinear algebra $\widetilde{W}$ obtained from $W_{N+2}^{(N+1)}$ via the Hamiltonian reduction constraints (3.2)-(3.3).

Let $\mathcal{J}$ be a current corresponding to the Cartan element t_0 of $sl(2)$ subalgebra. With respect to the adjoint action of t_0 the $sl(N)$ algebra can be decomposed into eigenspaces of t_0 with positive,null and negative eigenvalues h_a

$$sl(N) = (sl(N))_- \oplus (sl(N))_0 \oplus (sl(N))_+ \equiv \bigoplus_{h_a} (sl(N))_{h_a} \quad . \tag{3.4}$$

(In this subsection, the latin indices (a, b) run over the whole $sl(N)$, Greek indices (α, β) run over $(sl(N))_-$ and the barred Greek ones $(\bar{\alpha}, \bar{\beta})$ over $(sl(N))_0 \oplus (sl(N))_+$.) The Hamiltonian reduction associated with the embedding (3.4) can be performed by putting the appropriate constraints

$$J_\alpha - \chi_\alpha = 0 \quad , \quad \chi_\alpha \equiv \chi(J_\alpha) \tag{3.5}$$

on the currents J_α from $(sl(N))_-$ [2,7]. These constraints are the first class for integral gradings[6], which means that BRST formalism can be used.

In order to impose the constraints (3.5) in the framework of BRST approach one can introduce the fermionic ghost–anti-ghost pairs (b_α, c^α) with ghost numbers -1 and 1, respectively, for each current with the negative eigenvalues h_α:

$$c^\alpha(z_1) b_\beta(z_2) = \frac{\delta^\alpha_\beta}{z_{12}} \quad , \tag{3.6}$$

and the BRST charge

$$Q_{BRST} = \int dz J_{BRST}(z) = \int dz \left((J_\alpha - \chi(J_\alpha)) c^\alpha - \frac{1}{2} f^\gamma_{\alpha,\beta} b_\gamma c^\alpha c^\beta \right) \ , \tag{3.7}$$

which coincides with that given in the paper [14]. The currents of the algebra $Q\widetilde{SCA}{}_N^{lin}$ and the ghost fields b_α, c^α form the BRST complex, graded by the ghost number. The W algebra is defined in this approach as the algebra of operators generating the null cohomology of the BRST charge of this complex.

Following [14], let us introduce the "hatted" currents $\widehat{J}_a$:

$$\widehat{J}_a = \widetilde{J}_a + \sum_{\beta,\gamma} f^\gamma_{a,\beta} b_\gamma c^\beta \ , \tag{3.8}$$

where $f^\gamma_{a,\beta}$ are structure constants of $sl(N)$ in the basis (3.4). As shown in [14], the W-algebras, associated with the reductions of the affine $sl(N)$ can be embedded into linear algebras formed by the currents $\widehat{J}_{\overline{\alpha}}$. In contrast to the sl(N) algebra, our algebra $Q\widetilde{SCA}{}_N^{lin}$ contains, besides the $sl(N)$ currents, three additional ones $\widetilde{T}, \widetilde{U}, \widetilde{\overline{G}}{}^a$. Fortunately, the presence of these currents create no new problems while we construct a linearizing algebra for the reduction of $Q\widetilde{SCA}{}_N^{lin}$ by the BRST charge (3.7). Namely, the improved stress-tensor $\widehat{T}$ with respect to which J_{BRST} in eq. (3.7) is a spin 1 primary current can be easily constructed

$$\widehat{T} = \widetilde{T} + \mathcal{J}' + \sum_\alpha \{ -(1 + h_\alpha) b_\alpha c^{\alpha\prime} - h_\alpha b'_\alpha c^\alpha \} \ , \tag{3.9}$$

and so it belongs, together with $\widetilde{U}$, which commutes with Q_{BRST}, to a linear algebra we are searching for. As regards the current $\widetilde{\overline{G}}{}^i$, one could check that it extends the complex generated by the currents $\widehat{J}_a, b_\alpha, c^\beta$ with preserving the structure of the BRST subcomplexes of the paper [14], and forms, together with non-constrained currents $\widehat{J}_{\overline{\alpha}}$ and c^α, a reduced BRST subcomplex and subalgebra which do not contains the currents with negative ghost numbers. Hence, like in ref. [14], the W algebra closes not only modulo BRST exact operators, but it also closes in its

[6]Let us remind, that the half-integer gradings can be replaced by integer ones, leading to the same reduction [2].

own right. So, it is evident that the currents $\widehat{J}_{\overline{\alpha}}$ also will be present among the currents of the linearizing algebra in our case, as well as the currents $\widetilde{\widetilde{G}}^i$.

Thus, the set of currents $\widehat{T}$, $\widehat{J}_{\overline{\alpha}}$ (3.8),(3.9) and the currents

$$\widehat{U} \equiv \widetilde{U} \quad , \quad \widehat{\widetilde{G}} \equiv \widetilde{\widetilde{G}}^i \tag{3.10}$$

form the linear algebra W^{lin} for the nonlinear algebra W obtained from $W_{N+2}^{(N+1)}$ through the secondary Hamiltonian reduction associated with constraints (3.2)-(3.3).

3.2 Linearizing W_N algebras.

In this subsection we apply the general procedure described in the previous subsection to the case of the principal embedding of $sl(2)$ into $sl(N)$ algebra to construct the linear algebras W_N^{lin} which contain the nonlinear W_N algebras as subalgebras.

For the principal embedding of $sl(2)$ into $sl(N)$ with the currents $J_a^b, (1 \leq a, b \leq N, Tr(J) = 0)$, the current $\mathcal{J}$ is defined to be

$$\mathcal{J} = -\sum_{m=1}^{N-1} m J_{N-m}^{N-m} \quad , \tag{3.11}$$

and the decomposition of affine algebra $sl(N)$ reads as follows

$$\begin{aligned} (sl(N))_- &\propto \left\{ J_a^b, (2 \leq b \leq N, 1 \leq a < b) \right\} \\ (sl(N))_0 \oplus (sl(N))_+ &\propto \left\{ J_a^b, (1 \leq a \leq N-1, a \leq b \leq N) \right\} \quad , \end{aligned} \tag{3.12}$$

i.e. $(sl(N))_-$ consists of those entries of the $N \times N$ current matrix which stand below the main diagonal, and the remainder just constitutes the subalgebra $(sl(N))_0 \oplus (sl(N))_+$.

Now, using (2.9),(3.8) – (3.12), we are able to explicitly write the linear algebra W_{N+2}^{lin} which contains the W_{N+2} algebra as a subalgebra:

$$\begin{aligned}
\widehat{T}(z_1)\widehat{T}(z_2) &= \frac{(N+1)\left(1-(N+2)(N+3)\frac{(K-1)^2}{K}\right)}{2z_{12}^4} + \frac{2\widehat{T}}{z_{12}^2} + \frac{\widehat{T}'}{z_{12}} \quad , \\
\widehat{U}(z_1)\widehat{U}(z_2) &= \left(\frac{2NK}{2+N}\right)\frac{1}{z_{12}^2} \, , \\
\widehat{T}(z_1)\widehat{J}_a^b(z_2) &= \frac{(N+1-2a)(K-1)\delta_a^b}{z_{12}^3} + \frac{(b-a+1)\widehat{J}_a^b}{z_{12}^2} + \frac{\widehat{J}_a^{b\prime}}{z_{12}} \, , \\
\widehat{T}(z_1)\widehat{U}(z_2) &= -\frac{2N(K-1)}{z_{12}^3} + \frac{\widehat{U}}{z_{12}^2} + \frac{\widehat{U}'}{z_{12}} \, , \\
\widehat{T}(z_1)\widehat{\widetilde{G}}^i(z_2) &= \frac{(i+2)\widehat{\widetilde{G}}^i}{z_{12}^2} + \frac{\widehat{\widetilde{G}}^{i\prime}}{z_{12}} \, , \\
\widehat{J}_a^b(z_1)\widehat{J}_c^d(z_2) &= K\frac{\delta_a^d\delta_c^b - \frac{1}{N}\delta_a^b\delta_c^d}{z_{12}^2} + \frac{\delta_c^b\widehat{J}_a^d - \delta_a^d\widehat{J}_c^b}{z_{12}} \, , \\
\widehat{U}(z_1)\widehat{\widetilde{G}}^i(z_2) &= -\frac{\widehat{\widetilde{G}}^i}{z_{12}} \, , \; \widehat{J}_a^b(z_1)\widehat{\widetilde{G}}^i(z_2) = \frac{-\delta_a^i\widehat{\widetilde{G}}^b + \frac{1}{N}\delta_a^b\widehat{\widetilde{G}}^i}{z_{12}} \, , \\
\widehat{\widetilde{G}}^i(z_1)\widehat{\widetilde{G}}^j(z_2) &= \text{regular} \, ,
\end{aligned} \tag{3.13}$$

where the indices run over the following ranges:

$$\widehat{J}_a^b : (1 \le a \le N-1, a \le b \le N) \quad , \quad \widehat{\overline{G}}^i : (1 \le i \le N) \quad .$$

In this non-primary basis the currents $\widehat{\overline{G}}^i$ have the conformal weights $3, 4, ..., N+2$, and the stress-tensor $\widehat{T}$ coincides with the stress-tensor of W_{N+2} algebra.

It is also instructive to rewrite the W_{N+2}^{lin} algebra (3.13) in the primary basis $\left\{T, \widehat{U}, \widehat{J}_a^b, \widehat{\overline{G}}^i\right\}$, where a new stress-tensor T is defined as

$$T = \widehat{T} - \frac{(N+2)(K-1)}{2K}\widehat{U}' + \frac{K-1}{K}\sum_{m=1}^{N-1} m\left(\widehat{J}_{N-m}^{N-m}\right)' \tag{3.14}$$

and the OPE's have the following form

$$\begin{aligned}
T(z_1)T(z_2) &= \frac{N+1-6\frac{(K-1)^2}{K}}{2z_{12}^4} + \frac{2T}{z_{12}^2} + \frac{T'}{z_{12}} \quad , \quad \widehat{U}(z_1)\widehat{U}(z_2) = \left(\frac{2NK}{2+N}\right)\frac{1}{z_{12}^2} \, , \\
T(z_1)\widehat{J}_a^b(z_2) &= \frac{\left(1-\frac{a-b}{K}\right)\widehat{J}_a^b}{z_{12}^2} + \frac{\widehat{J}_a^{b\prime}}{z_{12}} \, , \\
T(z_1)\widehat{U}(z_2) &= \frac{\widehat{U}}{z_{12}^2} + \frac{\widehat{U}'}{z_{12}} \, , \\
T(z_1)\widehat{\overline{G}}^i(z_2) &= \frac{\left(\frac{3}{2}+\frac{1+2i}{2K}\right)\widehat{\overline{G}}^i}{z_{12}^2} + \frac{\widehat{\overline{G}}^{i\prime}}{z_{12}} \, , \\
\widehat{J}_a^b(z_1)\widehat{J}_c^d(z_2) &= K\frac{\delta_a^d\delta_c^b - \frac{1}{N}\delta_a^b\delta_c^d}{z_{12}^2} + \frac{\delta_c^b\widehat{J}_a^d - \delta_a^d\widehat{J}_c^b}{z_{12}} \, , \\
\widehat{U}(z_1)\widehat{\overline{G}}^i(z_2) &= -\frac{\widehat{\overline{G}}^i}{z_{12}} \, , \; \widehat{J}_a^b(z_1)\widehat{\overline{G}}^i(z_2) = \frac{-\delta_a^i\widehat{\overline{G}}^b + \frac{1}{N}\delta_a^b\widehat{\overline{G}}^i}{z_{12}} \, , \\
\widehat{\overline{G}}^i(z_1)\widehat{\overline{G}}^j(z_2) &= \text{regular} \, .
\end{aligned} \tag{3.15}$$

In this basis the "chain" structure of the algebras W_N^{lin} becomes most transparent. Namely, if we redefine the currents of W_{N+2}^{lin} as

$$\begin{aligned}
\mathcal{U}_1 &= \widehat{U} - N\sum_{m=1}^{N-1}\widehat{J}_m^m \, , \\
\mathcal{U} &= \frac{(N+2)(N-1)}{N(N+1)}\widehat{U} + \frac{2}{N+1}\sum_{m=1}^{N-1}\widehat{J}_m^m \\
\mathcal{T} &= T + \sqrt{\frac{N+2}{12KN^2(N+1)}}\mathcal{U}_1' \, , \quad \left(\text{or} \quad \mathcal{T} = T - \frac{N+2}{2KN^2(N+1)}(\mathcal{U}_1\mathcal{U}_1)\right) \, , \\
\mathcal{J}_a^b &= \widehat{J}_a^b - \frac{\delta_a^b}{N-1}\sum_{m=1}^{N-1}\widehat{J}_m^m \, , (1 \le a \le N-2, a \le b \le N-1) \, , \\
\mathcal{S}_a &= \widehat{J}_a^N \, , (1 \le a \le N-1) \, , \\
\overline{\mathcal{G}}^i &= \widehat{\overline{G}}^i \, , (1 \le i \le N-1) \, , \\
\overline{\mathcal{Q}} &= \widehat{\overline{G}}^N \, ,
\end{aligned} \tag{3.16}$$

then the subset $\mathcal{T},\mathcal{U},\mathcal{J}_a^b,\overline{\mathcal{G}}^i$ generates the algebra W_{N+1}^{lin} in the form (3.15). Thus, the W_{N+2}^{lin} algebras constructed have the following structure

$$W_{N+2}^{lin}=\left\{W_{N+1}^{lin},\mathcal{U}_1,\mathcal{S}_a,\overline{\mathcal{Q}}\right\} \tag{3.17}$$

and therefore there exists the following chain of embeddings

$$\ldots W_N^{lin}\subset W_{N+1}^{lin}\subset W_{N+2}^{lin}\ldots\quad . \tag{3.18}$$

Let us stress that the nonlinear W_{N+2} algebras do not possess the chain structure like (3.18), this property is inherent only to their linearizing algebras W_{N+2}^{lin}.

By this we finished the construction of linear algebras W_{N+2}^{lin} which contain W_{N+2} as subalgebras in a nonlinear basis. Let us repeat once more that the explicit expression for the transformations ¿from the currents of W_{N+2}^{lin} algebra to those forming W_{N+2} algebra is a matter of straightforward calculation once we know the exact structure of the linear algebra.

Finally, let us stress that knowing the structure of the linearized algebras W_{N+2}^{lin} helps us to reveal some interesting properties of the W_{N+2} algebras and their representations.

First of all, each realization of W_{N+2}^{lin} algebra gives rise to a realization of W_{N+2}. Hence, the relation between linear and nonlinear algebras opens a way to find new non-standard realizations of W_{N+2} algebras. As was shown in [5] for the particular case of W_3, these new realizations [3] can be useful for solving the problem of embedding Virasoro string into the W_3 one.

Among many interesting realizations of W_{N+2}^{lin} there is one very simple particular realization which can be described as follows. A careful inspection of the OPE's (3.15) shows that the currents

$$\widehat{\overline{G}}^i\ ,\ \widehat{J}_a^b:(1\leq a\leq N-1,a<b\leq N) \tag{3.19}$$

are null fields and so they can be consistently put equal to zero. In this case the algebra W_{N+2}^{lin} will contain only Virasoro stress tensor T and N $U(1)$-currents $\left\{\widehat{U},\widehat{J}_1^1,\ldots\widehat{J}_{N-1}^{N-1}\right\}$. Of course, there exists the basis, where all these currents commute with each other. The currents of W_{N+2} algebra are realized in this basis in terms of arbitrary stress tensor T_{Vir} with the central charge c_{Vir}

$$c_{Vir}=1-6\frac{(K-1)^2}{K} \tag{3.20}$$

and N decoupled commuting $U(1)$ currents. Surprisingly, the values of c_{Vir} corresponding to the minimal models of Virasoro algebra [15] at

$$K=\frac{p}{q}\Rightarrow c_{Vir}=1-6\frac{(p-q)^2}{pq} \tag{3.21}$$

induce the central charge $c_{W_{N+2}}$ of the minimal models for W_{N+2} algebra [10]

$$c_{W_{N+2}}=(N+1)\left(1-(N+2)(N+3)\frac{(p-q)^2}{pq}\right) \tag{3.22}$$

(let us remind that the stress tensor of W_{N+2} coincides with the stress tensor $\widehat{T}$ in the non-primary basis (3.13)). For the W_3 algebra this property has been discussed in [3].

3.3 Linearizing W_4 algebra.

In this subsection, as an example of our construction, we would like to present the explicit formulas concerning the linearization of W_4 algebra.

The structure of the linear algebra W_4^{lin} in the primary basis can be immediately read off from the OPE's (3.15) by putting $N = 2$. So, the algebra W_4^{lin} contains the currents $\left\{T, \widehat{U}, \widehat{J}_1^1, \widehat{J}_1^2, \widehat{\widehat{G}}{}^1, \widehat{\widehat{G}}{}^2\right\}$, with the conformal weights $\left\{2, 1, 1, \frac{K+1}{K}, \frac{3(K+1)}{2K}, \frac{3K+5}{2K}\right\}$, respectively.

Passing to the currents of W_4, goes over two steps.

Firstly, we must write down most general, nonlinear in the currents of W_4^{lin}, *invertible* expressions for the currents $\mathcal{T}_W, \mathcal{W}, \mathcal{V}$ with the desired conformal weights (2,3 and 4). It can be easily done in the nonprimary basis (3.13), where the stress tensor $\widehat{T}$ coincides with the stress tensor of W_4 algebra.

Secondly, we should calculate the OPE's between the constructed expressions and demand them to form a closed set.

This procedure completely fixes all coefficients in the expressions for the currents of W_4 algebra in the primary basis in terms of currents of W_4^{lin} (up to unessential rescalings). Let us stress that we do not need to know the explicit structure of W_4 algebra. By performing the second step, we automatically reconstruct the W_4 algebra.

Let us present here the results of our calculations for the W_4 algebra.

$$
\begin{aligned}
\mathcal{T}_W &= T + \frac{2(K-1)}{K}\widehat{U}' - \frac{K-1}{K}\widehat{J}_1^{1\prime} , \\
\mathcal{W} &= \widehat{\widehat{G}}{}^1 + \frac{K-1}{K}(T_1 - T_2)' + \frac{1}{K}\left((T_1 - T_2)\widehat{U}\right) - \frac{K-1}{K}\widehat{J}_1^{2\prime} - \frac{1}{K}(\widehat{J}_1^2\widehat{U}) , \\
\mathcal{V} &= -\widehat{\widehat{G}}{}^2 + \frac{K-1}{K}\widehat{\widehat{G}}{}^{1\prime} + \frac{1}{2K}\left((\widehat{J}_1^2\widehat{J}_1^2) + \widehat{J}_1^{2\prime}\right) + \frac{1}{K}\left((\widehat{U} - 2\widehat{J}_1^1)\widehat{\widehat{G}}{}^1\right) - \frac{1}{K}\left((T_1 - T_2)\widehat{J}_1^2\right) + \\
&\quad \frac{1}{2K}\left((T_1 - T_2)(T_1 - T_2)\right) - \frac{2}{K^2}\left(\widehat{J}_1^1\widehat{J}_1^2\right)' + \frac{1}{K^2}\left((T_1 + T_2)(2(K-1)\widehat{U}' + (\widehat{U}\widehat{U}))\right) + \\
&\quad \frac{K-1}{K^2}\left((T_1 + T_2)'\widehat{U}\right) + \frac{(K-1)^2}{2K^2}(T_1 + T_2)'' - \frac{(K-1)(2K-3)(3K-2)}{3K^3}\widehat{U}''' + \\
&\quad \frac{(3-2K)(3K-2)}{4K^3}(\widehat{U}''\widehat{U}) - \frac{16(6 - 13K + 6K^2)}{K(300 - 637K + 300K^2)}(\mathcal{T}_W\mathcal{T}_W) - \\
&\quad \frac{3(60 - 121K + 60K^2)(-6 + 13K - 6K^2)}{4(300 - 637K + 300K^2)}\mathcal{T}_W'' ,
\end{aligned} \tag{3.23}
$$

where the auxiliary currents T_1 and T_2 are defined as

$$
\begin{aligned}
T_1 &= T - \frac{1}{K}(\widehat{J}_1^1\widehat{J}_1^1) - \frac{1}{2K}(\widehat{U}\widehat{U}) , \\
T_2 &= \frac{1}{K}(\widehat{J}_1^1\widehat{J}_1^1) - \frac{K-1}{K}\widehat{J}_1^{1\prime} .
\end{aligned} \tag{3.24}
$$

For the W_4^{lin} algebra (3.15) the currents $\widehat{\widehat{G}}{}^1, \widehat{\widehat{G}}{}^2$ and $\widehat{J}_1^2$ are null-fields. So we can consistently put them equal to zero. In this case the expressions (3.23) provide us with the Miura realization of W_4 algebra in terms of two currents with conformal spins 2 (T_1, T_2) and with the same central charges, and one current with spin 1 $(\widehat{U})$ which commute with each other.

4 Conclusion.

In this letter we have constructed the linear (super)conformal algebras with finite numbers of generating currents which contain in some nonlinear basis a wide class of W-(super)algebras, including $W_N^{(N-1)}$, $U(N)$-superconformal as well as W_N nonlinear algebras. For the W_N algebras we do not have a rigorous proof of our conjecture about the general structure of the linearizing algebras, but we have shown that it works both for classical algebras (on the level of Poisson brackets) and some simplest examples of quantum algebras (e.g., for W_3, W_4). The explicit construction of the linearizing algebras W_{N+2}^{lin} for W_{N+2} reveals their many interesting properties: they have a "chain" structure (i.e. the linear algebras with a given N are subalgebras of those with a higher N), the central charge of the Virasoro subsector of these linear algebras in the parametrization corresponding to the Virasoro minimal models, while putting the null-fields equal to zero, induces the central charge for the minimal models of W_N, etc. This is the reasons why we believe that our conjecture is true.

It is interesting to note that, as we have explicitly demonstrated in the case of W_4 algebra, we do not need to know beforehand the structure relations of the nonlinear algebras, which rapidly become very complicated with growth of spins of the involved currents. Once we have constructed the linearizing algebra, we could algorithmically reproduce the structure of the corresponding nonlinear one. So, one of the main open questions now is how much information about the properties of a given nonlinear algebra we can extract from its linearizing algebra. The answer to this question could be important for applications of linearizing algebras to W-strings, integrable systems with W-type symmetry, etc. A detailed discussion of this issue will be given elsewhere.

Note Added.

After this paper was completed, we learned of a paper by J.O. Madsen and E. Ragoucy [16], which has some overlap with our work. They showed that the wide class of W-algebras (including W_n ones) can be linearized in the framework of the secondary hamiltonian reduction. However, they did not obtain the explicit expressions for the linearizing algebras (excepting W_4 case). The linearization of the (quasi)superconformal algebras was not considered, because their method does not allow fields with negative conformal weights.

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